Probabilistic Methods in Combinatorics

Instructor: Oliver Janzer

Assignment 13

To solve for the Example class on 27th May. Submit the solution of Problem 3 by Sunday 25th May if you wish feedback on it. Some hints will be given on Friday 23rd May.

Starred problems are typically harder. Don't worry if you cannot solve them.

Problem 1. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq \{0,1\}^N$ be all decreasing or all increasing families and let $\mathbb{P}$ be a product probability space on $\{0,1\}^N$. Then,

$$\mathbb{P}[\mathcal{F}_1 \cap \mathcal{F}_2 \cdots \cap \mathcal{F}_k] \geq \prod_{i=1}^{k} \mathbb{P}[\mathcal{F}_i].$$

Problem 2. Let G be a graph with m edges, and let S be a random set of vertices of G obtained by picking each vertex independently with probability $1/2$. Prove that the probability that S is an independent set in G is at least $(3/4)^m$.

Problem 3. A family of subsets $\mathcal{F}$ is called *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting families of subsets of $[n] := \{1, \ldots, n\}$. Show that $|\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k| \leq 2^n - 2^{n-k}$.

Problem 4. Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k-1$ is at least $1/4^k$.

Problem 5*. Let $S_1, \ldots, S_k$ be random subsets of $\{1, \ldots, n\}$, where each set S_i contains an element $x \in \{1, \ldots, n\}$ with probability $1/\sqrt{n}$ and all of these choices are independent. Prove that with probability at least $(1-1/e)^{\binom{k}{2}}$, we have for every $1 \leq i < j \leq k$, $S_i \cap S_j \neq \emptyset$.